Table 1 Values of A as a function of the electron energy in argon and hydrogen

	Electron energy, ev				
	30	50	100	150	200
Argon	0.45	0.9	1.6	2.2	...
Hydrogen	...	...	1.4	...	2.8

estimated by use of the parameter A, whose maximum value corresponds to the number of ionizing collisions made by an electron of a given energy when all the energy is expended in the gas. This number has been measured experimentally for different gases.[4]

For energies above 200 ev, A is almost proportional to the energy of the electron.

One can show[5, 6] that, in a reflex discharge, A is slightly lower than the maximum value given in Table 1. Typical values of the discharge voltage are 150 or 200 v. With $A = 2$ and $K = 0.1$, it is found that $\eta_E = 0.6$.

No experiments have been performed on a semitoroidal discharge, but the operation of a complete toroidal reflex discharge using oxide cathodes identical to those shown in Fig. 1 has been observed experimentally.[8, 9] Plasma densities attained in the toroidal discharge and in the linear discharge, using the same type of cathodes,[5–7] are of the order of 5×10^{13} electrons/cm^3 for a discharge current of 100 amp. Ionization rates are higher than 50%.

Efficiency will increase with higher discharge voltage. Discharge currents of 60 amps with voltage of 500 v have been obtained recently on a toroidal discharge with good stability. The cathode was operating with reduced emission. For this voltage, one would expect the value of A to be of the order of $A \simeq 8$.[4] For $A = 8$, $K = 0.1$, $\eta_E = 0.80$.

Discussion

A semitoroidal reflex discharge might have an efficiency comparable to that of an ionic propulsion device. Although the good operation of a toroidal reflex discharge has already been established, some questions remain unanswered.

1) The ions can be attracted by the negative potential of the cathode and made to fall on it. Therefore, K could be somewhat higher than S'/S.

2) It has been assumed that the losses are only due to ions falling to the cathodes. It is not sure that the losses in the torus, by perpendicular diffusion across the magnetic field and recombination on the wall, can be reduced to an insignificant amount. However, an encouraging result obtained on the toroidal discharge is the fact that the rate of loss measured experimentally[10] is in good agreement with the calculated value of loss rate taking into account classical diffusion due to collisions and drift losses to the curvature of the magnetic field. The drift losses can be at least partially eliminated by use of a bumpy magnetic field.[8] Anomalous diffusion does not seem to occur in our discharge for magnetic field up to 600 gauss (maximum available). However, anomalous diffusion occurs in a linear discharge for magnetic fields higher than 1000 gauss[7] and might occur also in the toroidal discharge. But high magnetic fields are not desirable for propulsion device for other reasons, i.e., loss of power in the coils and increase of weight.

3) What will be the ratio of useful propellant to propellant consumed? It is difficult to predict a priori, but it seems that the efficiency of ionization of the electrons in the reflex discharges that we have studied is at least as good as in those described by Kaufman and Reader[11] in which ratios up to 0.8 (80%) were obtained.

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Derivation of Element Stiffness Matrices by Assumed Stress Distributions

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IN a recent note,[1] the author has outlined a procedure for evaluating the element stiffness matrices to be used in connection with the displacement method of matrix structural analyses. The procedure is based on the representation of the displacement of a structural element in terms of displacement functions of m undetermined coefficients. In general, in order to satisfy the equilibrium of stress in the interior, m should be larger than the number of the generalized displacements n. When m and n are equal, these undetermined coefficients can be directly related to the generalized displacements. When m is larger than n, they can be evaluated by the employment of the principle of minimum potential energy. An important requirement is that the displacement functions must maintain compatibility with the adjacent elements. This condition has been emphasized by many previous authors.[2, 3]

This approach of assumed displacement functions is particularly suitable for one-dimensional elements, such as segments of axisymmetrical shells,[4] for which the displacement compatibility with the adjacent elements is completely satisfied when the corresponding generalized displacements coincide at the nodes. For two-dimensional problems such as general shells or plates under bending or in plane stress conditions, it is not always a straightforward matter to write down a displacement function that will yield compatible boundary displacements. For example, Melosh[2] has presented an expression for the bending displacement of a rectangular plate. That expression only provides continuity of displacements at all edges, but it will not maintain continuity of slopes along the normal directions of the four edges. The present author feels that a completely compatible boundary displacement for bending of plates and shells should include the slope continuity as well.

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This paper presents an alternative derivation of the element stiffness matrices. In this method, instead of a required continuous displacement function over the element, it is necessary only to write down the boundary displacements that will guarantee a complete displacement compatibility. The derivation is based on the principle of minimum complementary energy. The present proposed method is apparently different from the variational methods suggested by Melosh[2] and by Best.[5]

One approach to determining the stiffness matrix of a structural element consists of expressing the strain energy U in the element in terms of the generalized displacements $\{q\}$:

$$U = \tfrac{1}{2}\lfloor q \rfloor [k] \{q\} \tag{1}$$

and $[k]$ is the element stiffness matrix.

In writing an expression for U, either assumed continuous displacement functions or equilibrating stress distributions can be made.[6] It appears, however, that for an optimum choice both displacement compatibility and stress equilibrium conditions should be satisfied. This means that, prior to the evaluation of the stiffness matrix, the following problem in elasticity should be solved: given a solid with prescribed displacements at the boundary A_2, it is desirable to determine the stress distribution over the element. This problem can be solved by the principle of minimum complementary energy,[7] which may be stated as

$$\pi_c = U - \int_{A_2} u_i S_i \, dA = \min \tag{2}$$

where U is the strain energy in terms of stress components σ_{ij}, u_i is the component of the prescribed displacement, and S_i is the component of surface force which is related to the stress components by

$$S_i = \sigma_{ij} n_j \tag{3}$$

where n_j is the direction cosine of the surface normal.

In the application of this variational principle, one begins by expressing the stress distribution $\{\sigma\}$ $(= \{\sigma_{11}, \sigma_{22} \ldots\})$ in terms of m undetermined stress coefficients $\{\beta\}$ as follows:

$$\{\sigma\} = [P]\{\beta\} \tag{4}$$

where the terms of the matrix $[P]$ are functions of the coordinates $x_i (i = 1, 2, 3)$. The number of elements in $\{\beta\}$ is unlimited. When the stress-strain relations

$$\{\varepsilon\} = [N]\{\sigma\} \tag{5}$$

are introduced, one can express the internal strain energy as

$$U = \frac{1}{2}\int_V \lfloor \sigma \rfloor [N] \{\sigma\} dV \tag{6}$$

or

$$U = \tfrac{1}{2}\lfloor \beta \rfloor [H] \{\beta\} \tag{7}$$

where

$$[H] = \int_V [P^T][N][P] dV \tag{8}$$

It is seen that $[H]$ is symmetrical.

The prescribed displacements at the boundary A_2 are given in terms of the n generalized displacements $\{q\}$ at the nodes in the form of

$$\{u\} = [L]\{q\} \tag{9}$$

where the terms in the matrix $[L]$ contain coordinates on the surface. The surface force $\{S\}$ can be expressed in terms of the stresses $\{\sigma\}$ by means of Eq. (3) and hence can be related to the undetermined stress coefficients $\{\beta\}$, i.e.,

$$\{S\} = [R]\{\beta\} \tag{10}$$

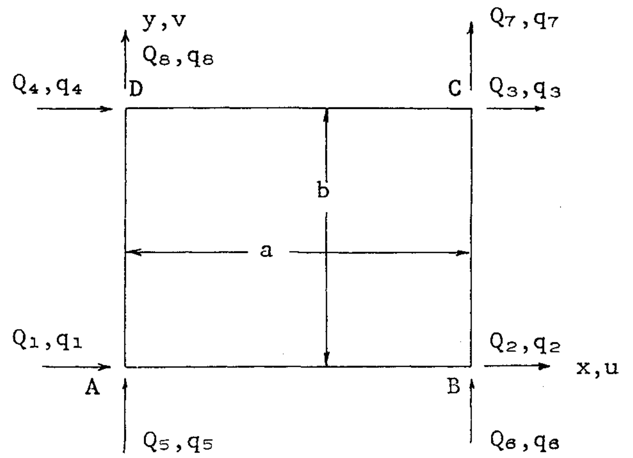

Fig. 1 Generalized forces and displacements of a rectangular element.

where the terms in $[R]$ also contain the coordinates on the surface.

The total complementary energy is then

$$\pi_c = \tfrac{1}{2}\lfloor \beta \rfloor [H]\{\beta\} - \lfloor \beta \rfloor [T]\{q\} \tag{11}$$

where

$$[T] = \int_{A_2} [R^T][L] dA \tag{12}$$

The condition of minimum complementary energy, i.e., $\partial\pi_c/\partial\beta_i = 0 (i = 1, \ldots m)$, yields

$$[H]\{\beta\} = [T]\{q\} \tag{13}$$

and

$$\{\beta\} = [H^{-1}][T]\{q\} \tag{14}$$

Substituting $\{\beta\}$ into Eq. (7), one obtains

$$U = \tfrac{1}{2}\lfloor q \rfloor [T^T][H^{-1}][T]\{q\} \tag{15}$$

By comparison with Eq. (1), one can conclude

$$[k] = [T^T][H^{-1}][T] \tag{16}$$

It is seen that the corresponding column of generalized forces $\{Q\}$ is

$$\{Q\} = [k]\{q\} \tag{17}$$

Substituting $[k]$ from Eq. (16) and then using Eq. (14), one finds

$$\{Q\} = [T^T]\{\beta\} \tag{18}$$

The matrix $[T^T]$ thus relates the equivalent generalized forces $\{Q\}$ and the assumed stress coefficients $\{\beta\}$. The physical interpretation here is that the work done by the generalized forces due to the corner displacements is equal to the work done by the boundary stresses due to the prescribed boundary displacements. This fact has already been discussed by Melosh.[2] Equation (12) indicates that there is a unique way of expressing the generalized forces in terms of the assumed stress functions. It happens that, for plane stress problems for which the edge displacements are assumed linear, the $[T^T]$ matrix obtained using Eq. (12) is exactly the same as that obtained by the usual procedure of lumping edge stresses according to the inverse ratio of the distances to the two edges. For the plate bending problem, however, a distribution of the edge shear forces according to the inverse ratio would be incorrect.

One very interesting observation is that a formula given by Best[8] and by Gallagher[6] for determining element stiffness matrices is of the form

$$[k] = [V][H^{-1}][V^T] \tag{19}$$

where $[H]$ is defined in the same way as Eq. (8), and $[V]$ is precisely the matrix relating the equivalent forces $\{Q\}$ and the stress coefficients $\{\beta\}$. Two important points, however, can be learned by following the development of the present paper: 1) there is a unique way for defining the equivalent generalized forces, and 2) the number of the assumed stress coefficients in $\{\beta\}$ need not be limited to $m = n - l$, where n is the number of the generalized displacement and l the number of rigid body degrees of freedom of the problem. It is apparent that there is a converging solution when n is sufficiently large.

To illustrate the proposed procedure, the stiffness matrix is determined of a rectangular plate (Fig. 1) under plane stress conditions. The eight generalized forces and the corresponding displacements are shown in the figure. The stress functions that satisfy the equation of equilibrium are as follows:

$$\left.\begin{aligned} \sigma_x &= \beta_1 + \beta_2 y + \beta_6 x + \beta_8 y^2 + \beta_{10} x^2 + \ldots \\ \sigma_y &= \beta_3 + \beta_4 x + \beta_7 y + \beta_9 x^2 + \beta_{10} y^2 + \ldots \\ \tau &= \beta_5 - \beta_6 y - \beta_7 x - 2\beta_{10} xy + \ldots \end{aligned}\right\} \tag{20}$$

The boundary force matrix $\{S\}$ consists of eight elements representing the x and y components of the boundary forces along the four edges. They are

$$\{S\} = \begin{bmatrix} (S_x)_{AB} \\ (S_y)_{AB} \\ (S_x)_{BC} \\ (S_y)_{BC} \\ (S_x)_{DC} \\ (S_y)_{DC} \\ (S_x)_{AD} \\ (S_y)_{CD} \end{bmatrix} = \begin{bmatrix} -\tau_{AB}(x) \\ -\sigma_{yAB}(x) \\ \sigma_{xBC}(y) \\ \tau_{BC}(y) \\ \tau_{DC}(x) \\ \sigma_{yDC}(x) \\ -\sigma_{xAD}(y) \\ -\tau_{AD}(y) \end{bmatrix} \tag{21}$$

For the case where the terms up to β_5 are retained, the $\{S\}$ matrix can be expressed in terms of $\{\beta\}$ as follows:

$$\{S\} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -x & 0 \\ 1 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & x & 0 \\ -1 & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \tag{22}$$

Here the matrix preceding $\{\beta\}$ is the $[R]$ matrix.

The corresponding boundary displacement matrix is

$$\{u\} = \begin{bmatrix} u_{AB}(x) \\ v_{AB}(x) \\ u_{BC}(y) \\ v_{BC}(y) \\ u_{DC}(x) \\ v_{DC}(x) \\ u_{AD}(y) \\ v_{AD}(y) \end{bmatrix} = \begin{bmatrix} 1-\frac{x}{a} & \frac{x}{a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{x}{a} & \frac{x}{a} & 0 & 0 \\ 0 & 1-\frac{y}{b} & \frac{y}{b} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\frac{y}{b} & \frac{y}{b} & 0 \\ 0 & 0 & \frac{x}{a} & 1-\frac{x}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{a} & 1-\frac{x}{a} \\ 1-\frac{y}{b} & 0 & 0 & \frac{y}{b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{y}{b} & 0 & 0 & \frac{y}{b} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} \tag{23}$$

Here the $\{q\}$ matrix is premultiplied by the $[L]$ matrix.

The $[T]$ matrix obtained by Eq. (12) is as follows:

$$[T] = \begin{bmatrix} -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & 0 & 0 & 0 & 0 \\ -\frac{b^2}{6} & \frac{b^2}{6} & \frac{b^2}{3} & -\frac{b^2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ 0 & 0 & 0 & 0 & -\frac{a^2}{6} & -\frac{a^2}{3} & \frac{a^2}{3} & \frac{a^2}{6} \\ -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} & -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} \end{bmatrix} \tag{24}$$

As an example, the stiffness matrix has been evaluated of a square element with thickness equal to t and Poisson's ratio equal to $\frac{1}{3}$. When terms up to β_5 are chosen from Eq. (18), the $[k]$ matrix obtained by Eq. (16) is

$$Et \begin{bmatrix} 0.45833 & -0.27083 & -0.29167 & 0.10417 & 0.18750 & 0.00000 & -0.18750 & 0.00000 \\ & 0.45833 & 0.10417 & -0.29167 & 0.00000 & -0.18750 & 0.00000 & 0.18750 \\ & & 0.45833 & -0.27083 & -0.18750 & 0.00000 & 0.18750 & 0.00000 \\ & & & 0.45833 & 0.00000 & 0.18750 & 0.00000 & -0.18750 \\ & & & & 0.45833 & 0.10417 & -0.29167 & -0.27083 \\ & \text{symmetric} & & & & 0.45833 & -0.27083 & -0.29167 \\ & & & & & & 0.45833 & 0.10417 \\ & & & & & & & 0.45833 \end{bmatrix}$$

Stiffness matrices have also been calculated by taking terms up to β_7, β_9, and β_{10}, respectively. It turns out that, when only five significant figures are kept, the three resulting matrices are identical, indicating that the convergency of the solution is indeed very rapid. This resulting matrix is given in the following:

$$
Et\begin{bmatrix}
0.46875 & -0.28125 & -0.28125 & 0.09375 & 0.18750 & 0.00000 & -0.18750 & 0.00000 \\
 & 0.46875 & 0.09375 & -0.28125 & 0.00000 & -0.18750 & 0.00000 & 0.18750 \\
 & & 0.46875 & -0.28125 & -0.18750 & 0.00000 & 0.18750 & 0.00000 \\
 & & & 0.46875 & 0.00000 & 0.18750 & 0.00000 & -0.18750 \\
 & & & & 0.46875 & 0.09375 & -0.28125 & -0.28125 \\
 & \text{symmetric} & & & & 0.46875 & -0.28125 & -0.28125 \\
 & & & & & & 0.46875 & 0.09375 \\
 & & & & & & & 0.46875
\end{bmatrix}
$$

Finally, it should be mentioned that the foregoing converging value of the element stiffness matrix can also be obtained by employing the principle of minimum potential energy[1] if a sufficiently large number of terms of the following displacement functions are used:

$$
\begin{aligned}
u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + xy(x - a)(y - b)(\alpha_9 + \alpha_{11} x + \alpha_{13} y + \ldots) \\
v &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy + xy(x - a)(y - b)(\alpha_{10} + \alpha_{12} y + \alpha_{14} x + \ldots)
\end{aligned} \tag{25}
$$

For example, when terms up to α_8 are used, the resulting square element stiffness matrix by Eq. (15) of Ref. 1 is as follows:

$$
Et\begin{bmatrix}
0.5 & -0.3125 & -0.25 & 0.0625 & 0.1875 & 0 & -0.1875 & 0 \\
 & 0.5 & 0.0625 & -0.25 & 0 & -0.1875 & 0 & 0.1875 \\
 & & 0.5 & -0.3125 & -0.1875 & 0 & 0.1875 & 0 \\
 & & & 0.5 & 0 & 0.1875 & 0 & -0.1875 \\
 & & & & 0.5 & 0.0625 & -0.25 & -0.3125 \\
 & \text{symmetric} & & & & 0.5 & -0.3125 & -0.25 \\
 & & & & & & 0.5 & 0.0625 \\
 & & & & & & & 0.5
\end{bmatrix}
$$

When terms up to α_{10} are used, the resulting matrix is

$$
Et\begin{bmatrix}
0.47396 & -0.28646 & -0.27604 & 0.08854 & 0.1857 & 0 & -0.1875 & 0 \\
 & 0.47396 & 0.08854 & -0.27604 & 0 & -0.1875 & 0 & 0.1875 \\
 & & 0.47396 & -0.28646 & -0.1875 & 0 & 0.1875 & 0 \\
 & & & 0.47396 & 0 & 0.1875 & 0 & -0.1875 \\
 & & & & 0.47396 & 0.08854 & -0.27604 & -0.28646 \\
 & \text{symmetric} & & & & 0.47396 & -0.28646 & -0.27604 \\
 & & & & & & 0.47396 & 0.08854 \\
 & & & & & & & 0.47396
\end{bmatrix}
$$

It is seen that the result is converging, although not as rapidly as that by the method of assumed stress functions.

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A Simple Derivation of Three-Dimensional Characteristic Relations

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THE method of characteristics for three-dimensional flow has been studied by a number of investigators.[1–5] Most of the existing derivations of the conditions for characteristic surfaces and the compatibility relations seem either general and lengthy or particular and sketchy. This note presents a simple, straightforward derivation of the characteristic surfaces and compatibility relations of steady, inviscid, isoenergetic, three-dimensional flow, based on a basic idea of characteristics.

The fundamental equations in the present case are

$$
(u^2 - a^2)u_x + (v^2 - a^2)v_y + (w^2 - a^2)w_z + uv(v_x + u_y) + vw(w_y + v_z) + wu(u_z + w_x) = 0 \tag{1}
$$

$$
v(v_x - u_y) - w(u_z - w_x) + a^2 s_x' = 0 \tag{2}
$$

$$
w(w_y - v_z) - u(v_x - u_y) + a^2 s_y' = 0 \tag{3}
$$

$$
u(u_z - w_x) - v(w_y - v_z) + a^2 s_z' = 0 \tag{4}
$$

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